Imperial College London

Lecture 7

Systems & Laplace Transform

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10 things you have learned about signals (1)

1. Signals can be represented in **time domain** or **frequency domain**.
2. Any signal can be made up from **weighted sum of sinusoidal** signals.
3. A sinusoid at frequency ω and amplitude A can be an everlasting sine wave (A sin ωt), cosine wave (Acos ωt) or exponential ($A/2$ $e^{j\omega t}$). Furthermore, two sinusoids at different frequencies have **NOTHING in common**.
4. For a **time-limited** signal, moving between time and frequency domain is done through **Fourier Transform**.
5. A **periodic signal** is represented in the frequency domain in **Fourier series**, where the fundamental frequency f_0 is 1/period of the signal, and all the other frequency are integer multiple of f_0.

10 things you have learned about signals (2)

6. You must sample a signal at a sampling frequency f_s which is **at least twice** that of the maximum signal frequency f_{max}: $f_s \geq 2*f_{max}$.

7. When **sampling signal** at f_s, the **spectrum** of the original signal is **repeated** at EVERY multiple of sampling frequency, i.e $\pm nf_s$, n = 1, 2, 3...

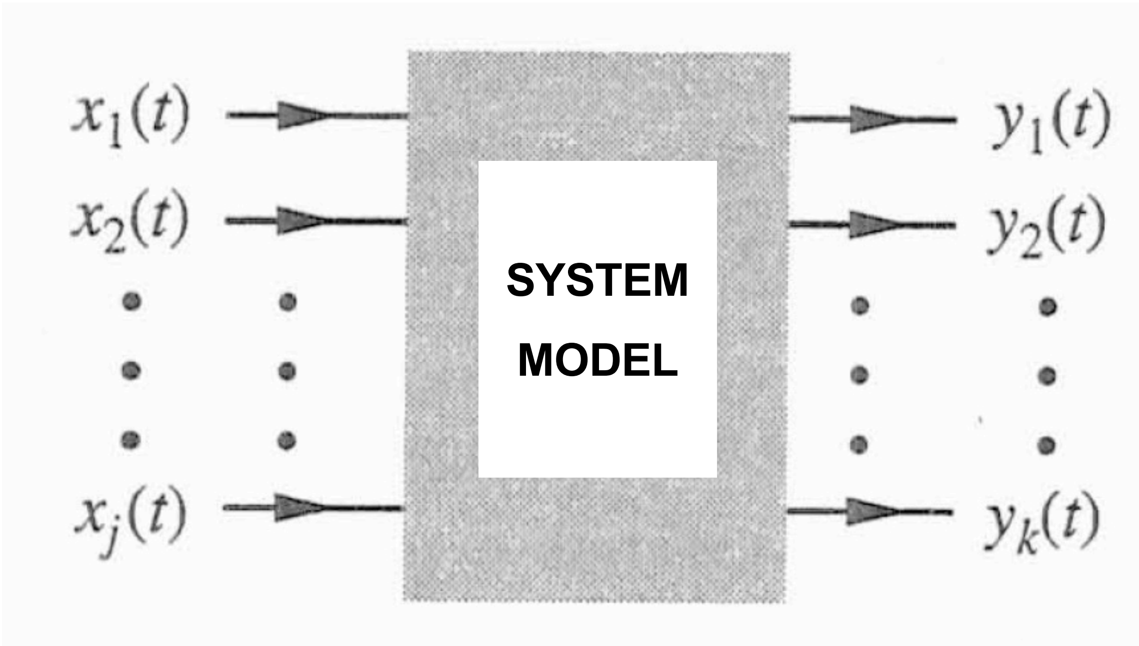
8. If you sample a signal which has a frequency component higher than fs/2, **aliasing** occurs (which results in **spectral folding**).

9. When you **extract** a portion of a signal, you effectively multiply the signal with a **rectangular window**, which results in spreading of energy to neigbouring frequency components. This is known as "**leakage**".

10. You can **reduce** this **leakage** by multiplying your signal with a **special window** function which has smooth instead of shape edges.

What are Systems?

- Systems are used to **process signals** to **modify** or **extract information**
- Physical systems – characterized by their **input-output relationships**
- E.g. electrical systems are characterized by voltage-current relationships for components and the **laws of interconnections** (i.e. Kirchhoff's laws)
- From this, we derive a **mathematical model** of the system
- "**Black box**" model of a system:



L1.6

Linear Systems (1)

- A **linear system** exhibits the **additivity** property:

if $x_1 \longrightarrow y_1 \quad x_2 \longrightarrow y_2$ **then** $x_1 + x_2 \longrightarrow y_1 + y_2$

- It also must satisfy the **homogeneity** or **scaling** property:

if $x \longrightarrow y$ **then** $kx \longrightarrow ky$

- These can be combined into the property of **superposition**:

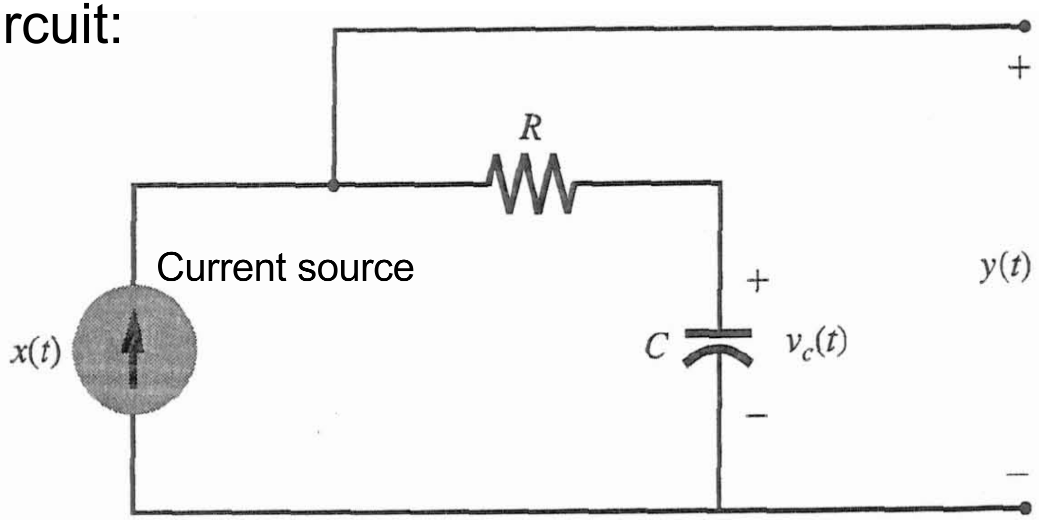
if $x_1 \longrightarrow y_1 \quad x_2 \longrightarrow y_2$ **then** $k_1x_1 + k_2x_2 \longrightarrow k_1y_1 + k_2y_2$

- A non-linear system is one that is NOT linear (i.e. does not obey the principle of superposition)

L1.7-1

Linear Systems (2)

- Consider the following simple RC circuit:



- Output *y(t)* relates to *x(t)* by:

$$y(t) = Rx(t) + \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

- The second term can be expanded:

$$y(t) = Rx(t) + \frac{1}{C}\int_{-\infty}^{0} x(\tau)\,d\tau + \frac{1}{C}\int_{0}^{t} x(\tau)\,d\tau$$

$$y(t) = v_C(0) + Rx(t) + \frac{1}{C}\int_{0}^{t} x(\tau)\,d\tau$$

- This is a **single-input, single-output** (SISO) system. In general, a system can be multiple-input, multiple-output (**MIMO**).

L1.6

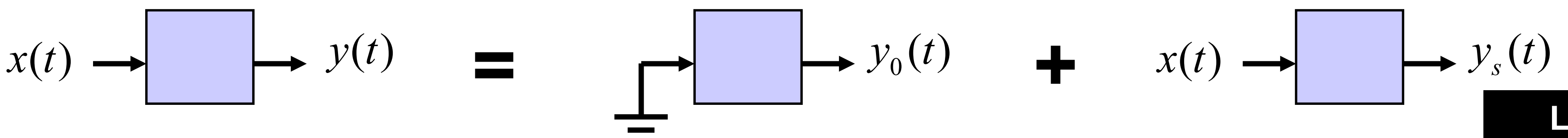
Linear Systems (3)

- A system's output for t ≥ 0 is result of 2 independent causes:
  1. Initial conditions when t = 0 (**zero-input response**)
  2. Input x(t) for t ≥ 0 (**zero-state response**)

- Decomposition property:

Total response = zero-input response + zero-state response

$$y(t) = \underbrace{v_C(0)}_{\textbf{zero-input response}} + \underbrace{Rx(t) + \frac{1}{C}\int_0^t x(\tau)\,d\tau}_{\textbf{zero-state response}} \qquad t \geq 0$$



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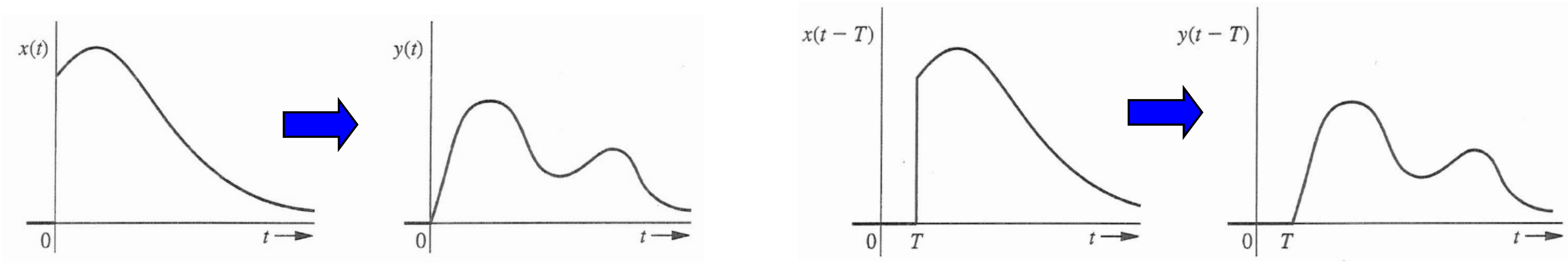
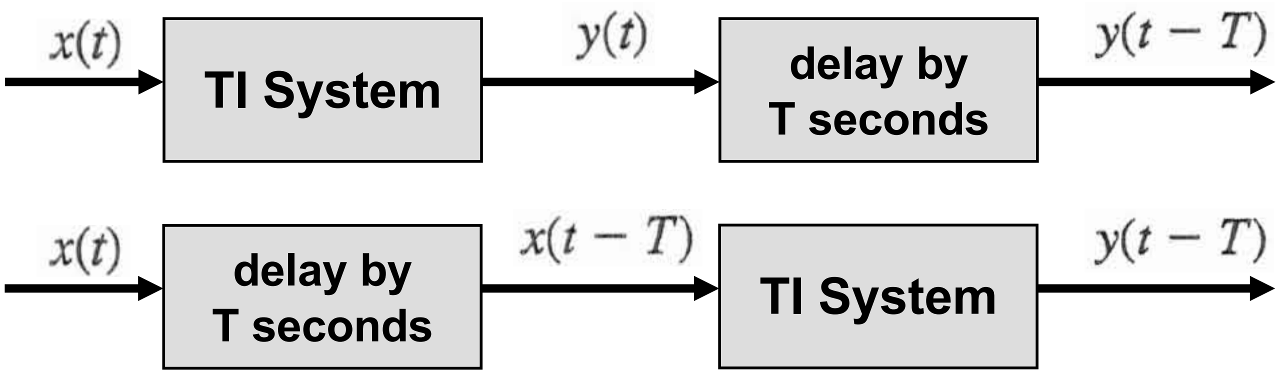
Time-Invariant Systems

- **Time-invariant system** is one whose parameters do not change with time:



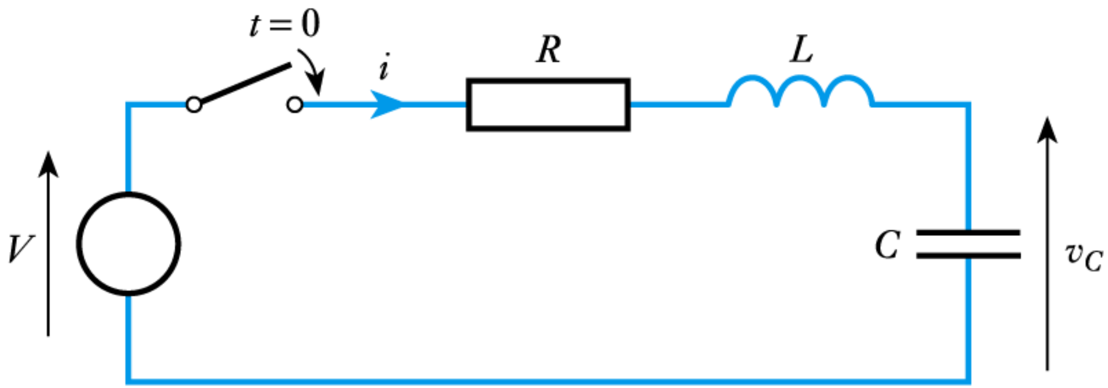


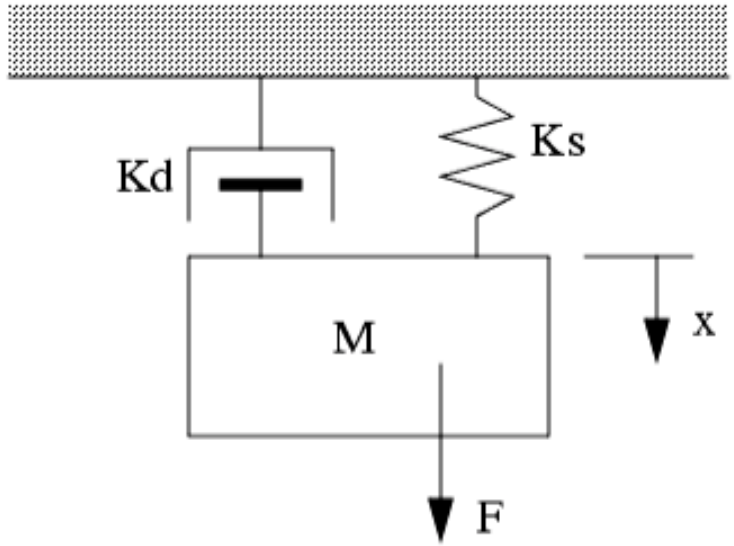
- Linear time-invariant (**LTI**) systems – main type of systems for this course.

L1.7-2

System modelling using ODEs

- Many systems in electrical and mechanical engineering where input and output are related by **ordinary differential equations (ODEs)**
- For example:





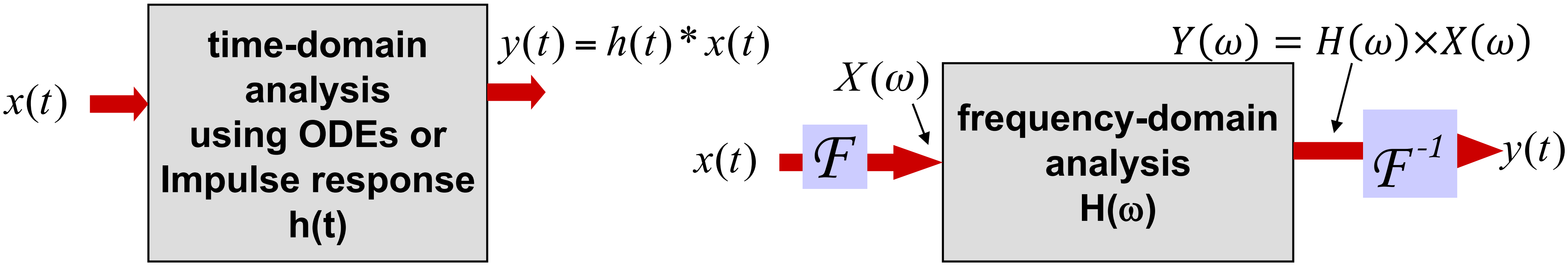
$$v_L(t) + v_R(t) + v_c(t) = V$$

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = V$$

$$M\ \ddot{x}(t) + K_d\dot{x}(t) + K_s x(t) = F(t)$$

L1.8

System Analysis in time and frequency domains



- Analyse system using differential equations or using the system's **impulse response** h(t) (later lecture)
- Analyse system using **frequency response** H(ω)
- Analyse system behaviour in time-domain via solving differential equations can be tedious.
- Could use impulse response and **convolution** (later topic), but could be expensive.
- Using Fourier transforms and frequency response to analyse (and predict behaviour of) a system has limitations.
- Frequency response is only useful in predicting **steady-state behaviour** of a system, not transient behaviour.
- Alternative – use Laplace transform to transform both system and signals to the complex Laplace variable, the s-domain.

Laplace Transform (1)

- Laplace Transform is a method that converts differential equations in time-domain into algebraic equations in complex Laplace variable s-domain.
- Definition of Laplace Transform $\mathcal{L}$ is:

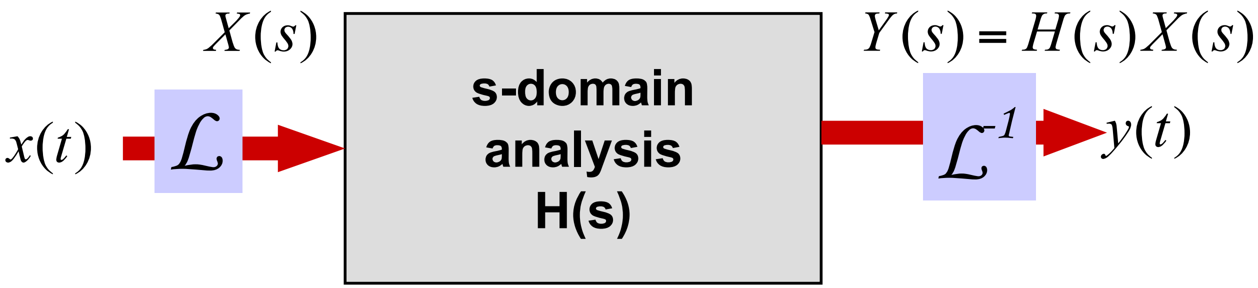
$$\mathcal{L}[X(t) = X(s) = \int_0^{\infty} x(t)e^{-st}dt$$

Fourier Transform

$$\mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$s = \alpha + j\omega$$

- Once transformed to the s-domain, analysis and prediction of the system becomes easy if we know the system's characteristic H(s), which is also called the **transfer function** (more later)



L4.1

Laplace Transform (2)

- Laplace Transform obeys laws of **linearity**:

$$\mathcal{L}\left[\beta_1 x_1(t) + \beta_2 x_2(t)\right] = \beta_1 \mathcal{L}[x_1(t)] + \beta_2 \mathcal{L}[x_2(t)]$$

- The Laplace transform of **an impulse function**:

$$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = 1 \qquad \text{for all } s$$

$$\boxed{\mathcal{L}[\delta(t)] \quad \Leftrightarrow \quad 1}$$

- The Laplace transform of a **unit step function**:

$$\mathcal{L}[u(t)] = \int_0^\infty u(t) e^{-st} dt = \int_0^\infty e^{-st} dt$$

$$= -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad \operatorname{Re} s > 0$$

$$\boxed{\mathcal{L}[u(t)] \quad \Leftrightarrow \quad \frac{1}{s}}$$

L4.1

Laplace Transform (3)

- Laplace Transform of $e^{at}\,u(t)$:

$$\mathcal{L}[e^{at}u(t)] = \int_0^{\infty} e^{at}e^{-st}\,dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\,dt = \frac{1}{s-a}$$

$$\mathcal{L}[e^{at}u(t)] \Leftrightarrow \frac{1}{s-a}$$

- Laplace Transform of $\cos\omega_0 t\; u(t)$:

$$\mathcal{L}[\cos\omega_0 t\; u(t)] = \frac{1}{2}\mathcal{L}[e^{j\omega_0 t}u(t) + e^{-j\omega_0 t}u(t)]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] = \frac{s}{s^2+\omega_0^{\,2}}$$

$$\mathcal{L}[\cos\omega_0 t u(t)] \Leftrightarrow \frac{s}{s^2+\omega_0^{\,2}}$$

L4.1

Laplace Transform (4)

- Laplace Transform of a **differentiator** $\dot{x}(t) = \dfrac{dx(t)}{dt}$:

$$\mathcal{L}[\frac{dx(t)}{dt}] = \int_{t=0}^{\infty} \frac{dx(t)}{dt} e^{-st}\, dt$$

- It can be shown (using integration by parts) that this result in:

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0)$$

- If x(0) = 0 (i.e. zero initial condition), then $\mathcal{L}[\dot{x}(t)] = sX(s)$

- Therefore, differentiation in the time domain is multiplication by s in the s-domain:

$$\frac{d}{dt} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad s$$

Laplace Transform (5)

- Laplace Transform of an **integrator** $\int_{\tau=0}^{t} x(\tau)d\tau$:

$$\text{Let } g(t) = \int_{\tau=0}^{t} x(\tau)d\tau$$

$$\text{then } x(t) = \frac{dg(t)}{dt}, \quad \text{and} \quad g(0) = 0$$

- From last slide

$$\mathcal{L}[x(t)] = \mathcal{L}[\dot{g}(t)] = sG(s) - g(0) = sG(s)$$

- Therefore

$$\mathcal{L}[g(t)] = \frac{1}{s}X(s)$$

- Therefore, integration in the time domain is multiplication by 1/s in the s-domain:

$$\int_{t=0}^{t} \quad \xleftrightarrow{\mathcal{L}} \quad s^{-1}$$

Laplace transform Pairs (1)

- Finding inverse Laplace transform requires integration in the complex plane – beyond scope of this course.
- So, use a Laplace transform table (analogous to the Fourier Transform table).

No.	$x(t)$	$X(s)$
* 1	$\delta(t)$	1
* 2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$

L4.1

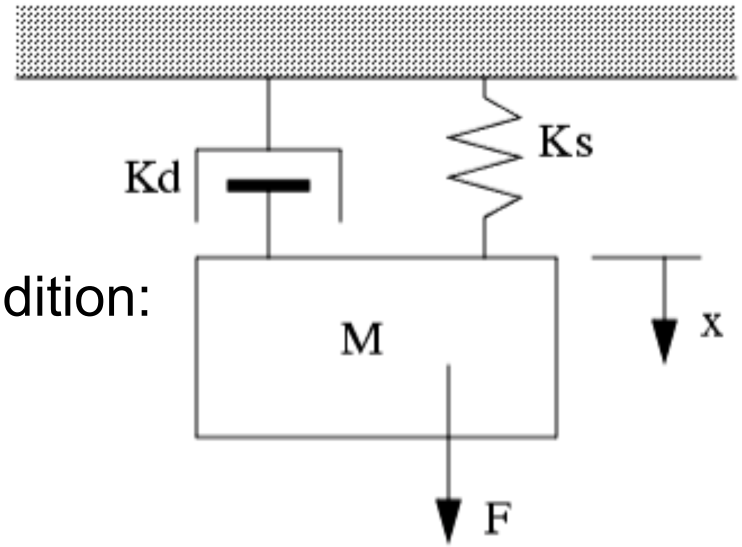
Laplace transform Pairs (2)

	No.	$x(t)$	$X(s)$
*	5	$e^{\lambda t}u(t)$	$\dfrac{1}{s-\lambda}$
	6	$te^{\lambda t}u(t)$	$\dfrac{1}{(s-\lambda)^2}$
	7	$t^n e^{\lambda t}u(t)$	$\dfrac{n!}{(s-\lambda)^{n+1}}$
*	8a	$\cos bt\, u(t)$	$\dfrac{s}{s^2+b^2}$
*	8b	$\sin bt\, u(t)$	$\dfrac{b}{s^2+b^2}$
*	9a	$e^{-at}\cos bt\, u(t)$	$\dfrac{s+a}{(s+a)^2+b^2}$
*	9b	$e^{-at}\sin bt\, u(t)$	$\dfrac{b}{(s+a)^2+b^2}$

L4.1

Laplace Transform vs Differential Equations

- Since $$\mathcal{L}\left[\frac{x(t)}{dt}\right] = sX(s)$$
  we can generalise higher order differential as: $$\frac{d^k}{dt^k} \overset{\mathcal{L}}{\leftrightarrow} s^k$$
- Therefore, consider the mechanical system in slide 10:



$$M\ \ddot{x}(t) + K_d \dot{x}(t) + K_s x(t) = F(t)$$

- Apply Laplace transform assuming zero initial condition:

$$Ms^2 X(s) + K_d sX(s) + K_s X(s) = F(s)$$

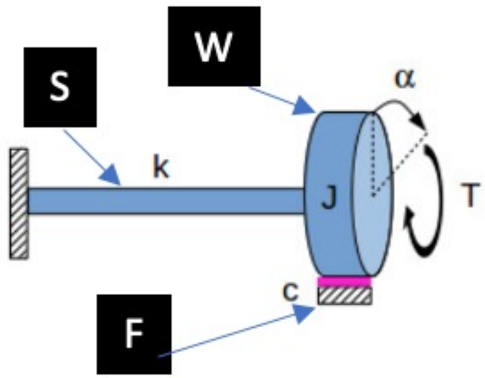
$$(Ms^2 + K_d s + K_s)X(s) = F(s)$$

$$\Longrightarrow H(s) = \frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + K_d s + K_s)}$$

H(s) is TRANSFER FUNCTION

Using Laplace Transform to model a system

- Here is another mechanical system with a wheel (taken from last year's examination paper):



T = external torque on the wheel
α = angle of rotation of the wheel
J = moment of inertia
k = shaft stiffness
c = damping coefficent

- The relationship between the wheel angle α and the external torque T is given by the following equation:

$$T - k\alpha - c\frac{d\alpha}{dt} - J\frac{d^2\alpha}{dt^2} = 0$$

- Apply Laplace transform assuming zero initial condition:

$$T(s) - k\alpha(s) - cs\alpha(s) - Js^2\alpha(s) = 0$$

Hence,

$$H(s) = \frac{\alpha(s)}{T(s)} = \frac{1}{Js^2 + cs + k}$$

Three Big Ideas

1. Laplace transform is useful for analysing systems. It maps time domain behaviour to the complex frequency s-domain where $s = \alpha + j\omega$. This contrasts with Fourier transform which maps to frequency (or ω) domain.

2. Laplace transform converts mathematical models of real systems described using differential equations in time domain to algebraic equation in s-domain. This is possible because:

$$\mathcal{L}\left(\frac{d}{dt}\right) = s \quad \text{and} \quad \mathcal{L}\left(\frac{d^2}{dt^2}\right) = s^2$$

3. Transfer function of a system H(s) is the Laplace transform of the output signal Y(s) divided by the Laplace transform of the input signal X(s):

$$H(s) = \frac{Output\ Y(s)}{Input\ X(s)}$$